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Shear viscosity of the A_1-phase of superfluid ^{3}He

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Abstract
The scattering processes between the quasiparticles in the spin-up superfluid and the quasiparticles in the spin-down normal fluid are added to the other relevant scattering processes in the Boltzmann collision terms. The Boltzmann equation has been solved exactly for temperatures just below T_{c1}. The shear viscosity component of the A_1-phase drops as $C_1(1 - \frac{T}{T_{c1}})^{1/2}$. The numerical factor C_1 is in fairly good agreement with the experiments.

1. Introduction

The viscosity of the A_1-phase of superfluid ^{3}He is investigated at temperatures close to the transition temperature, T_{c1}, where the maximum gap in the excitation spectrum is small in comparison with the thermal energy k_BT. It is supposed that in the A_1-phase only spin-up pairs exist, since it follows from the free energy expression that below the transition temperature, T_{c1}, it is lowered by formation of spin-up pairs. Formation of spin-down pairs becomes favourable below a temperature T_{c2} [1].

A number of viscosity measurements have been carried out in the A_1 and A_2 phases of superfluid ^{3}He at different high magnetic fields [2, 3]. The previous viscosity measurements were determined in low magnetic fields [4]. All the results show a sharp decrease proportional to the opening of the superfluid energy gap, $\Delta \propto (1 - \frac{T}{T_{c1}})^{1/2}$ near T_{c1} and goes as T^{-2} for very low temperatures. The calculations of the former region are under construction and will be published elsewhere. Obtaining the values of viscosity coefficient on the whole region of temperatures is not analytically possible.

The viscosity of the A-phase, in zero magnetic field, has been calculated exactly for temperatures close to T_c by Bhattacharyya *et al* [5] and Pethick *et al* [6], and the viscosity in the presence of a magnetic field has been considered by Shahzamanian [7]. The results of Bhattacharyya *et al* and Pethick *et al* on the viscosity drop as $(1 - \frac{T}{T_c})^{1/2}$ for temperatures close to T_c and the exact coefficient of $(1 - \frac{T}{T_c})^{1/2}$ has been expressed as a function of normal state properties.

In this paper we use the Boltzmann equation approach to obtain the viscosity of the A_1-phase for temperatures just below T_{c1}. One, therefore, needs to derive the correct form of the Boltzmann equation. This problem in the A-phase has been discussed by Bhattacharyya *et al* [5] and Pethick *et al* [6] extensively. The streaming terms in the Boltzmann equation have the standard form, but the collision term is more complicated than the normal state. In a normal Fermi liquid at low temperatures the only important collision process is the scattering of pairs of quasiparticles, but in a superfluid the quasiparticle number is not conserved, so one also has to take into account decay processes in which a single quasiparticle decays into three, and the inverse processes, in which three quasiparticles coalesce to form one. In the A_1-phase one has to take into account other processes which come from the scattering between superfluid quasiparticles in the spin-up population, the so-called Bogoliubov quasiparticles, and the normal fluid quasiparticles in the spin-down population. We shall evaluate the collision probabilities for these new processes.

The Boltzmann equation for a normal Fermi liquid has been solved exactly [8, 9]. This equation has been also solved exactly for temperatures close to T_c by Bhattacharyya *et al* [5]. The difference between the collision terms for the normal and superfluid states was treated as a perturbation. We use their method to solve the Boltzmann equation for the A_1-phase.

The paper is organized as follows. In section 2 by writing the interaction between the quasiparticles we obtain the collision terms for all the processes. Section 3 is allocated to writing the Boltzmann equation and its solution for the A_1-phase, then the shear viscosity tensor is calculated for temperatures close to T_{c1}. Finally in section 4 we give some remarks and concluding results.

2. Collision integral

To obtain the collision integral, we start with the interaction between the quasiparticles in the spin-up superfluid and spin-down normal fluid. This will be found by performing a Bogoliubov transformation on the normal-state interaction. The Bogoliubov transformation between the normal quasiparticle creation and annihilation operators $a^+_{\vec{p},\sigma}$ and $a_{\vec{p},\sigma}$ and the creation and annihilation operators $\alpha^+_{\vec{p},\sigma}$ and $\alpha_{\vec{p},\sigma}$ in the superfluid may be written as

$$\begin{aligned} a_{\vec{p},\sigma} &= u(\vec{p})_{\sigma\sigma'}\alpha_{\vec{p},\sigma'} - \upsilon(\vec{p})_{\sigma\sigma'}\alpha^+_{-\vec{p},\sigma'} \\ a^+_{-\vec{p},\sigma} &= \upsilon^*(\vec{p})_{\sigma\sigma'}\alpha_{\vec{p},\sigma'} + u(\vec{p})_{\sigma\sigma'}\alpha^+_{-\vec{p},\sigma'}. \end{aligned} \tag{1}$$

For the non-unitary state of the A_1-phase, we have the following properties between u and υ [10].

$$u(-\vec{P})_{\uparrow\uparrow} = u(\vec{P})_{\uparrow\uparrow}, \qquad \text{and} \qquad \upsilon(-\vec{P})_{\uparrow\uparrow} = -\upsilon(\vec{P})_{\uparrow\uparrow}. \tag{2}$$

The normal-state interaction is

$$H = \frac{1}{4}\sum_{1,2,3,4} \langle 3,4|T|1,2\rangle a^+_4 a^+_3 a_1 a_2 \tag{3}$$

where $i = 1, 2, 3$ and 4 stands for both momentum ($\vec{P}_i$) and spin (σ_i) variables. By using (1) in (3) the interaction between quasiparticles in the A_1-phase of superfluid is

$$\begin{aligned} H = \frac{1}{4}\sum_{\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4} &\{T_t[-\upsilon^*(\vec{P}_4)_{\uparrow\uparrow}\alpha_{\vec{P}_4\uparrow} - u(\vec{P}_4)_{\uparrow\uparrow}\alpha^+_{\vec{P}_4\uparrow}] \\ &\times [-\upsilon^*(\vec{P}_3)_{\uparrow\uparrow}\alpha_{-\vec{P}_3\uparrow} - u(\vec{P}_3)_{\uparrow\uparrow}\alpha^+_{\vec{P}_3\uparrow}][u(\vec{P}_1)_{\uparrow\uparrow}\alpha_{\vec{p}_1\uparrow} - \upsilon(\vec{P}_1)_{\uparrow\uparrow}\alpha^+_{-\vec{p}_1\uparrow}] \\ &\times [u(\vec{P}_2)_{\uparrow\uparrow}\alpha_{\vec{P}_2\uparrow} - \upsilon(\vec{P}_2)_{\uparrow\uparrow}\alpha^+_{-\vec{p}_2\uparrow}] + \frac{1}{2}(T_t - T_s)[-\upsilon^*(\vec{P}_4)_{\uparrow\uparrow}\alpha_{-\vec{P}_4\uparrow} \\ &- u(\vec{P}_4)_{\uparrow\uparrow}\alpha^+_{\vec{P}_4\uparrow}]\alpha^+_{\vec{P}_3\downarrow}[u(\vec{P}_1)_{\uparrow\uparrow}\alpha_{\vec{P}_1\uparrow} - \upsilon(\vec{P}_1)_{\uparrow\uparrow}\alpha^+_{-\vec{p}_1\uparrow}]\alpha_{\vec{P}_2\downarrow} + \frac{1}{2}(T_t + T_s)\alpha^+_{\vec{P}_4\downarrow} \\ &\times [-\upsilon^*(\vec{P}_3)_{\uparrow\uparrow}\alpha_{-\vec{P}_3\uparrow} - u(\vec{P}_3)_{\uparrow\uparrow}\alpha^+_{\vec{P}_3\uparrow}][u(\vec{P}_1)_{\uparrow\uparrow}\alpha_{\vec{P}_1\uparrow} - \upsilon(\vec{P}_1)_{\uparrow\uparrow}\alpha_{-\vec{P}_1\uparrow}]\alpha_{\vec{P}_2\downarrow}\} \end{aligned} \tag{4}$$

where the amplitudes are given by

$$\langle\uparrow\uparrow|T|\uparrow\uparrow\rangle \equiv T_t, \quad \langle\uparrow\downarrow|T|\uparrow\downarrow\rangle \equiv \tfrac{1}{2}(T_t + T_s) \quad \text{and} \quad \langle\uparrow\downarrow|T|\downarrow\uparrow\rangle \equiv \tfrac{1}{2}(T_t - T_s). \tag{5}$$

The gap parameter of the non-unitary state of the A_1-phase, $\Delta_{\vec{P}}$, has the same $\bar{P}$-dependence as the A-phase, i.e. it has the axial structure. Furthermore in the A_1-phase we may write $E_{\vec{P}}^2 = \varepsilon_{\vec{P}}^2 + |\Delta_{\vec{P}\uparrow\uparrow}|^2$, where $\varepsilon_{\vec{P}}$ is the normal-state quasiparticle energy measured with respect to the chemical potential and $\Delta_{\vec{P}\uparrow\uparrow}$ is the magnitude of the gap in the direction $\vec{P}$ on the Fermi surface [10]. For the discussion of collision processes in the A_1-phase at temperatures just below the transition temperature it is most convenient to work in terms of quasiparticles which are related as closely as possible to the quasiparticle in the normal state. Accordingly we take the quasiparticle energy to be

$$E_{\vec{P}\uparrow} = (\varepsilon_{\vec{P}\uparrow}^2 + \Delta_{\vec{P}\uparrow\uparrow}^2)^{1/2}\,\mathrm{sgn}\,\varepsilon_{\vec{P}}. \tag{6}$$

Since we are interested only in changes in the collision integral of order $\Delta_{\vec{P}}$, we need, therefore, to retain only terms involving no more than a single $\upsilon_{\vec{P}}$ factor. As we have mentioned previously, in a superfluid the quasiparticle number is not conserved, and therefore scattering processes other than those in a normal Fermi liquid can occur. The first term in equation (4) indicates the scattering processes similar to those in the A-phase, i.e. two, decay and coalescence processes, which have been considered extensively by Bhattacharyya *et al* [5]. Here for brevity we write the final results of the collision terms corresponding to the first term in equation (4)

$$\left(\frac{\partial n_1}{\partial t}\right)_{coll} = -\sum_{\vec{P}_2,\vec{P}_3,\vec{P}_4} \frac{2\pi}{\hbar}\frac{1}{4}|T_t|^2 n_1 n_2(1-n_3)(1-n_4)\delta_{\vec{P}_1+\vec{P}_2,\vec{P}_3+\vec{P}_4}\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \times [\Psi_1 + f_1(f_2\Psi_2 - f_3\Psi_3 - f_4\Psi_4)] \tag{7}$$

where n_i is the quasiparticle distribution function, Ψ_i is the deviation function defined in terms of the local equilibrium distribution function $n_i^{l.e.}(E_i)$ by the relation

$$n_i = n_i^{l.e.}[1 - n_i^{l.e.}(E_i)]\Psi_i + n_i^{\ell.e.}(E_i) \tag{8}$$

and

$$f_{\vec{P}_i} = |u(\vec{P}_i)_{\uparrow\uparrow}|^2 - |\upsilon(\vec{P}_i)_{\uparrow\uparrow}|^2 = \frac{\varepsilon_{\vec{P}_i}}{E_{\vec{P}_i}} \equiv V_{\vec{P}_i}. \tag{9}$$

For the A_1-phase we may write

$$f_{\vec{P}_i} = V_{\vec{P}_i} = |\varepsilon_{\vec{P}_i}|/(\varepsilon_{\vec{P}_i}^2 + \Delta_{\uparrow\uparrow}^2\sin^2\theta)^{1/2} \tag{10}$$

where θ is the angle between $\hat{P}_i$ and the orbital anisotropy axis $\hat{l}$. The second and third terms in the interaction between the quasiparticles in equation (4) indicate the two quasiparticle scattering process and the coalescence scattering process between the Bogoliubov quasiparticles in the up-spin superfluid and the quasiparticles in the down-spin normal fluid. When the collision terms are linearized we have for the two-quasiparticle scattering

$$\left(\frac{\partial n_1}{\partial t}\right)_{coll} = -\sum_{\vec{P}_2,\vec{P}_3,\vec{P}_4} W'_{S,N}(1,2,3,4)[n_1 n_2(1-n_3)(1-n_4)(\Psi_1+\Psi_2-\Psi_3-\Psi_4)] \times \delta_{\vec{P}_1+\vec{P}_2,\vec{P}_3+\vec{P}_4}\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \tag{11}$$

where

$$W'_{S,N} = \frac{2\pi}{\hbar}\frac{1}{4}\left[\frac{1}{4}|T_t - T_S|^2|u_1^2|u_4|^2 + \frac{1}{4}|T_t + T_S|^2|u_1|^2|u_3|^2\right]. \tag{12}$$

Note that we have replaced the superfluid quasiparticle energy by the corresponding normal-state energy in appropriate places, since this does not affect a contribution of order $\Delta_{\vec{P}}$. For the process in which quasiparticles 1, 2 and -3 coalesce to give quasiparticle 4 the linearized collision term is

$$\left(\frac{\partial n_1}{\partial t}\right)_{coll} = -\sum_{\vec{P}_2,\vec{P}_3,\vec{P}_4} \frac{2\pi}{\hbar}\frac{1}{4}\{[|T_t - T_s|^2(|\upsilon_1|^2 + |\upsilon_3|^2)](n_1 n_2 n_{-3}(1-n_4)) \times \delta(E_1 + E_2 + E_{-3} - E_4)(\Psi_1 + \Psi_2 + \Psi_{-3} - \Psi_4) + [|T_t + T_S|^2(|\upsilon_1|^2 + |\upsilon_3|^2)] \times (n_1 n_2(1-n_3)n_{-4})\delta(E_1 + E_2 - E_3 + E_{-4}) \times (\Psi_1 + \Psi_2 - \Psi_3 + \Psi_{-4})\}\delta_{\vec{P}_1+\vec{P}_2,\vec{P}_3+\vec{P}_4}. \tag{13}$$

If we use the particle–hole symmetry of a degenerate Fermi system we may replace E_{-i} by $-E_i$, which has been used in obtaining equation (7) too. For the viscosity consideration, we may write $\Psi_2 = \Psi_{-2}(-E_2)$. The collision integral (13), hence, may be rewritten as

$$\left(\frac{\partial n_1}{\partial t}\right)_{coll} = -\sum_{\vec{P}_2,\vec{P}_3,\vec{P}_4} \frac{2\pi}{\hbar}\frac{1}{4}\frac{1}{4}\{[|T_t - T_S|^2(|\upsilon_4|^2 + |\upsilon_1|^2) + |T_t + T_S|^2 \times (|\upsilon_3|^2 + |\upsilon_1|^2)]n_1 n_2(1-n_3)(1-n_4)(\Psi_1 + \Psi_2 - \Psi_3 - \Psi_4)\}\delta_{\vec{P}_1+\vec{P}_2,\vec{P}_3+\vec{P}_4}. \tag{14}$$

Finally, by adding the collision terms in equations (11) and (14) we get

$$\left(\frac{\partial n_1}{\partial t}\right)_{coll} = -\sum_{\vec{P}_2,\vec{P}_3,\vec{P}_4} \frac{2\pi}{\hbar}\frac{1}{4} n_1 n_2(1-n_3)(1-n_4)\delta_{\vec{P}_1+\vec{P}_2,\vec{P}_3+\vec{P}_4} \times \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\{[\tfrac{3}{2}|T_t|^2 + \tfrac{1}{2}|T_S|^2]\Psi_1 + [f_1 f_2(\tfrac{3}{2}|T_t|^2 + \tfrac{1}{2}|T_S|^2) + \tfrac{1}{2}(1 - f_1 f_2)(|T_t|^2 + |T_S|^2)]\Psi_2 - [f_1 f_3(\tfrac{3}{2}|T_t|^2 + \tfrac{1}{2}|T_S|^2) + \tfrac{1}{2}(1 - f_1 f_3)(|T_t|^2 + |T_S|^2 - \tfrac{1}{2}|T_t + T_S|^2)]\Psi_3 - [f_1 f_4(\tfrac{3}{2}|T_t|^2 + \tfrac{1}{2}|T_S|^2) + \tfrac{1}{2}(1 - f_1 f_4)(|T_t|^2 + |T_S|^2 - \tfrac{1}{2}|T_t - T_S|^2)]\Psi_4\}; \tag{15}$$

one obtains the normal-state collision integral by putting $f_i = 1$ in equation (15).

3. Viscosity

Before writing a formula for the shear viscosity we write the Boltzmann equation for the A_1-phase

$$-(\vec{P}_1)_i(\vec{\upsilon}_1)_j \frac{\partial n_1}{\partial E_1}\left(\frac{\partial u_k}{\partial r_l} + \frac{\partial u_l}{\partial r_k}\right) = \left(\frac{\partial n_1}{\partial t}\right)_{coll} \tag{16}$$

where the collision integral operation is written in equation (15), and we pick out the terms in streaming terms of the Boltzmann equation which are relevant to viscosity. u_k is the k component of a spatially varying velocity $\vec{u}$. One would usually like to express Ψ_i in terms of the corresponding quantity for the normal state. For this purpose it is more convenient to work with the function $X_i \equiv \Psi_i/V_i$, since in the superfluid close to T_{c_1} it differs from the normal-state value by amounts of order $\Delta_{\vec{P}}/k_B T_c$.

Hence equation (16) becomes

$$\hat{P}_{1i}\hat{P}_{1j}\upsilon_F P_F \frac{\partial n_1}{\partial \varepsilon_1}\left(\frac{\partial u_k}{\partial r_l}+\frac{\partial u_l}{\partial r_k}\right) = \sum_{\vec{P}_2,\vec{P}_3,\vec{P}_4} \frac{2\pi}{\hbar}\frac{1}{4} n_l n_2(1-n_3)(1-n_4)$$
$$\times\, \delta_{\vec{P}_1+\vec{P}_2,\vec{P}_3+\vec{P}_4}\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)$$
$$\times\, \{[\tfrac{3}{2}|T_t|^2+\tfrac{1}{2}|T_s|^2]X_1 + [V_2^2(\tfrac{3}{2}|T_t|^2+\tfrac{1}{2}|T_S|^2)+\tfrac{1}{2}(1-V_2^2)(|T_t|^2+|T_S|^2)]$$
$$\times\, X_2 - [V_3^2(\tfrac{3}{2}|T_t|^2+\tfrac{1}{2}|T_S|^2)+\tfrac{1}{2}(1-V_3^2)(|T_t|^2+|T_S|^2-\tfrac{1}{2}|T_t+T_S|^2)]X_3$$
$$-\, [V_4^2(\tfrac{3}{2}|T_t|^2+\tfrac{1}{2}|T_S|^2)+\tfrac{1}{2}(1-V_4^2)(|T_t|^2+|T_S|^2-\tfrac{1}{2}|T_t-T_S|^2)]X_4\} \qquad (17)$$

where we put $(\frac{V_i}{V_1}-V_i^2)\cong(1-V_i^2)$, since it does not affect a contribution of order $\Delta_{\bar{P}}$.

Now the Boltzmann equation may be replaced by a one-dimensional integral equation. For this purpose we define the function $Q(\hat{P}_1,t_1)$ as

$$\Psi_1 = \frac{\upsilon_F V_1}{k_B T} P_F \tau_o 2\cosh\left(\frac{t_1}{2}\right)\left(\frac{\partial u_k}{\partial r_l}+\frac{\partial u_l}{\partial r_k}\right) Q(\hat{P}_1,t_1) \qquad (18)$$

where $t=\frac{\varepsilon}{k_B T}$. By substituting equation (18) into (17) we get

$$\frac{\hat{P}_{1i}\hat{P}_{1j}}{\cosh(t/2)} = (\pi^2+t^2)Q_{ij}(\hat{P}_1,t_1) - \int_{-\infty}^{\infty} \mathrm{d}t'\, F(t-t')[\alpha_2 V^2(\hat{P}_1,t')$$
$$+\, \beta_2(1-V^2(\hat{P}_1,t'))]Q_{ij}(\hat{P}_1,t') \qquad (19)$$

where $Q_{ij}(\hat{P},t)\equiv Q(\hat{P},t)\hat{P}_i\hat{P}_j$,

$$\alpha_2 = 2\langle W_N(\theta,\phi)[-P_2(\cos\theta_{12})+P_2(\cos\theta_{13})+P_2(\cos\theta_{14})]\rangle/\langle W_N(\theta,\phi)\rangle, \qquad (20)$$

$$\beta_2 = 2\left\langle \frac{2\pi}{\hbar}\frac{1}{4}\left[-\frac{1}{2}(|T_t|^2+|T_S|^2)P_2(\cos\theta_{12}) + \frac{1}{2}\left(|T_t|^2+|T_S|^2-\frac{1}{2}|T_s+T_t|^2\right)P_2(\cos\theta_{13})\right.\right.$$
$$\left.\left. +\, \frac{1}{2}\left(|T_t|^2+|T_S|^2-\frac{1}{2}|T_t-T_s|^2\right)P_2(\cos\theta_{14})\right]\right\rangle \Big/ \langle W_N(\theta,\phi)\rangle, \qquad (21)$$

θ_{ij} denotes the angle between $\vec{P}_i$ and $\vec{P}_j$,

$$\langle W_N(\theta,\phi)\rangle \equiv \int \frac{\mathrm{d}\Omega}{4\pi\cos(\theta/2)}\frac{2\pi}{\hbar}\frac{1}{8}(3|T_t|^2+|T_S|^2) \qquad (22)$$

and

$$F(t-t') \equiv \frac{t-t'}{2\sinh[(t-t')/2]}.$$

Bhattacharyya *et al* [5] by using the S- and P-wave approximation for the scattering amplitudes calculate the values of α_2 and $\langle W_N(\theta,\phi)\rangle$ for different values of pressure. Here we use the Pfitzner procedure [11] for calculating the values of β_2 and α_2 which appears in equation (19). It should be noted that the β_2-coefficient comes through the one-dimensional integral equation for the presence of the new scattering processes in the A_1-phase. We can use the quasiparticle scattering amplitude (QSA) of normal Fermi fluids instead of superfluid QSA for temperatures near $T_{c\uparrow}$. By using a general polynomial expansion of the QSA in normal Fermi fluids in equations (20)–(22), namely [11]

$$\upsilon(0)T_{s,t} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} a_{\ell k}X_{\ell k}(\upsilon,P) \qquad (\ell \text{ even, odd}) \qquad (23)$$

where the coefficients with ℓ even (odd) belong to the singlet (triplet) part of the QSA,

$$X_{\ell k}(v,P) = (k+1)^{1/2}(2\ell+1)^{1/2}(P^2/4-1)^{\ell} P_\ell(v) P_{k-\ell}^{(2\ell+1,0)}(P^2/2-1)$$
$$k = 0,1,\ldots,;\ \ell = 0,1,\ldots,k, \tag{24}$$

$$P = 2\cos\frac{\theta}{2}, \qquad v = \cos\phi. \tag{25}$$

Moreover, since we follow the procedure of Pfitzner [11] in calculating the QSA in the normal fluid at $T_{c\uparrow}$, we may truncate equation (23) at $k = 3$ for pressure 34.4 bar. By using the values of $a_{\ell k}$ from table 3 of [11] and performing numerically the integrals in equations (20) and (21), we get the following results:

$$\alpha_2 = 1.48 \qquad \text{and} \qquad \beta_2 = 0.72.$$

Now we write a formula for the shear viscosity. The momentum flux tensor may be written as

$$\prod\nolimits_{lm} = \sum_{\vec{P}} \vec{P}_l \left(\frac{\partial E_{\vec{P}}}{\partial \vec{P}_m}\right) \delta n_{\vec{P}} \tag{26}$$

where $\delta n_{\vec{P}} = n_{\vec{P}} - n_{\vec{P}}^{l.e.}$ characterize the deviation from local equilibrium, and from equation (8) we have

$$\delta\, n_{\vec{P}} = n_{\vec{P}}^{l.e.}(l - n_{\vec{P}}^{l.e.})\Psi_{\vec{P}}. \tag{27}$$

The shear viscosity is a fourth-rank tensor, which is defined by the relation

$$\prod\nolimits_{lm} = -\eta_{lmij}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right). \tag{28}$$

In writing the above equation we have supposed that $l \neq m$ and $i \neq j$. When equation (27), with consideration of equation (18), is substituted in equation (26) and then compared with equation (28), we get

$$\eta_{lmij} = 15\eta \langle\langle (V X_{lm}, V Q_{ij}) \rangle\rangle \equiv 15\eta Y_{lmij} \tag{29}$$

where $\eta = \frac{1}{5}\rho\frac{m^*}{m}v_F^2\tau_0$, τ_0 is the characteristic relaxation time and is given by

$$\tau_0 = \frac{8\pi^4\hbar^6}{m^{*3}(k_B T)^2\langle W_N\rangle}, \tag{30}$$

$$X_{lm} \equiv \frac{\hat{P}_l \hat{P}_m}{\cosh(t/2)}, \tag{31}$$

$$\langle\langle \cdots \rangle\rangle = \int \frac{\mathrm{d}\Omega_{\vec{P}}}{4\pi}, \ldots, \tag{32}$$

and

$$(A,B) = \int_{-\infty}^{\infty} A(t)B(t)\,\mathrm{d}t. \tag{33}$$

The kernel in the integral equation (19) has no structure on a scale $t \approx \frac{\Delta}{k_B T}$ and we may write equation (19) in the form

$$X_{ij} = (H_0 + H_1)Q_{ij} \tag{34}$$

where $H_o Q_{ij}$ is the right-hand side of equation (19) with $V = 1$, and

$$H_1 Q_{ij} \equiv \int_{-\infty}^{\infty} \mathrm{d}t'\, F(t-t')(1 - V^2(t'))(\alpha_2^2 - \beta_2^2) Q_{ij}(t') = \pi(\alpha_2 - \beta_2)\tilde{\Delta}_{\vec{P}} Q_{ij}(0) F(t). \tag{35}$$

In obtaining the last term in equation (35) we have used the following formula for any function $A(t)$ having no structure on a scale $\Delta/k_B T$:

$$\int_{-\infty}^{\infty} A(t)[1 - V^2(t)] = A(0)\int_{-\infty}^{\infty}[1 - V^2(t)]\,\mathrm{d}t = \pi A(0)\tilde{\Delta} \tag{36}$$

where $\tilde{\Delta} = \frac{\Delta}{k_B T}$. The dimensionless viscosity in equation (29) can be written as

$$Y_{lmij} = \langle\langle (X_{lm}, Q_{ij})\rangle\rangle - \langle\langle (X_{lm}, (1 - V^2)Q_{ij})\rangle\rangle = \langle\langle (X_{lm}, Q_{ij})\rangle\rangle - \pi\langle\langle \tilde{\Delta} X_{lm}(0), Q_{ij}(0)\rangle\rangle; \tag{37}$$

we write $Q_{ij} = Q_{0ij} + Q_{1ij}$, where Q_{0ij} is the unperturbed solution and $Q_{1ij} \propto \Delta_{\vec{P}}$ is the change due to the perturbation. By equating the terms independent of $\Delta_{\vec{P}}$ and those linear in $\Delta_{\vec{P}}$ to zero in equation (34), we have

$$\begin{aligned} &X_{ij} = H_0 Q_{0ij} \\ &0 = H_1 Q_{0ij} + H_0 Q_{1ij} \qquad \text{or} \qquad Q_{1ij} = -H_0^{-1} H_1 Q_{0ij} \end{aligned} \tag{38}$$

where the first equation is the normal-state Boltzmann equation. Hence to the lowest order in $\tilde{\Delta}$ we have

$$Y_{lmij} = \langle\langle (X_{lm}, Q_{0ij})\rangle\rangle - \langle\langle (X_{lm}, H_0^{-1} H_1 Q_{0ij})\rangle\rangle - \pi\langle\langle \tilde{\Delta} X_{lm}(0), Q_{ij}(0)\rangle\rangle. \tag{39}$$

The second term in the right-hand side of equation (39) can be written as

$$\langle\langle (X_{lm}, H_0^{-1} H_1 Q_{0ij})\rangle\rangle = \pi(\alpha_2 - \beta_2)\left\langle\!\!\left\langle \tilde{\Delta}_P Q_{0ij}(0)\int_{-\infty}^{\infty}\mathrm{d}t\, F(t) Q_{0lm}(t)\right\rangle\!\!\right\rangle. \tag{40}$$

The integral term in equation (40) can be obtained simply by putting $t = 0$, $V(t) = 1$ in equation (19). Finally we may write

$$\eta_{lmij} = \eta_{lmij}^n - 15\eta\langle\langle \hat{P}_l \hat{P}_m \hat{P}_i \hat{P}_j \tilde{\Delta}_{\vec{P}}\rangle\rangle\left[\pi Q_N(0)\frac{\beta_2}{\alpha_2} + \pi^3 Q_N^2(0)\left(1 - \frac{\beta_2}{\alpha_2}\right)\right] \tag{41}$$

where $\eta_{lmij}^n = \eta(\delta_{li}\delta_{mj} + \delta_{lj}\delta_{mi})(X, Q_N) = \eta Y_N$. The values of $Q_N(0)$ and Y_N have been evaluated by Bhattacharyya *et al* [5].

If the orbital axis is taken to be the Z axis, the shear viscosity has two different components, η_{xy} and $\eta_{zx} = \eta_{zy}$. The angular expressions in (41) for these components are

$$\begin{aligned} &\langle\langle P_x P_y P_x P_y \tilde{\Delta}_{\vec{P}}\rangle\rangle = (5\pi/256)\tilde{\Delta}_{\max} \\ &\langle\langle P_z P_x P_z P_x \tilde{\Delta}_{\vec{P}}\rangle\rangle = (\tfrac{4}{5})(5\pi/256)\tilde{\Delta}_{\max} \end{aligned} \tag{42}$$

where $\Delta_{\max}$ is the maximum value of the A_1-phase gap parameter. Formula (41) and (42) give

$$\frac{\delta\eta_{xy}}{\eta_{xy}} = -\left(\frac{75\pi}{256}\right)\left[\pi Q_N(0)\frac{\beta_2}{\alpha_2} + \pi^3 Q_N^2(0)\left(1 - \frac{\beta_2}{\alpha_2}\right)\right]\tilde{\Delta}_{\max}$$

and

$$\frac{\delta\eta_{zx}}{\eta_{zx}} = -\left(\frac{4}{5}\right)\left(\frac{75\pi}{256}\right)\left[\pi Q_N(0)\frac{\beta_2}{\alpha_2} + \pi^3 Q_N^2(0)\left(1 - \frac{\beta_2}{\alpha_2}\right)\right]\tilde{\Delta}_{\max}. \tag{43}$$

4. Conclusions and some remarks

To compare the results with the experiments [2–4] one has to know $\Delta_{\max}$ as a function of temperature. Pethick *et al* [6] by taking the spin averaged gap generalize their results of the A-phase to the A_1-phase. In weak-coupling theory one has $\Delta_{\max}(T) = (5/4)^{1/2} 3.06 k_B T_c (1 - \frac{T}{T_c})^{1/2}$ for the ABM state and, hence, the spin averaged gap in the A_1-phase is $\Delta_{\max} =$

$3.42[\frac{1}{2}k_B T_{C\uparrow}(1 - \frac{T}{T_{C\uparrow}})^{1/2}]$. By taking the strong-coupling effect into account, finally we have $\Delta_{\max} = 3.54[\frac{1}{2}k_B T_{C\uparrow}(1 - \frac{T}{T_{c\uparrow}})^{1/2}]$.

As we mentioned previously, these scattering processes between the quasiparticles in the up-spin superfluid and quasiparticles in the down-spin normal fluid play an important role in obtaining the Boltzmann equation for the A_1-phase.In this paper we take them into account and the results show themselves through the factor β_2 in equations (19) and (41).

The values of α_2 and β_2 depend slightly on the pressure through the Landau parameters. For pressures 21 bar and 34.36 bar, the melting pressure, the values of β_2 are respectively 0.79 and 0.72. The values of the last bracket in equation (43) for 21 and 34.36 bar pressures are respectively 2.27 and 2.12.

The viscosity data in the A_1- and A_2-phases of superfluid ^{3}He were analysed by Alvesola *et al* [4] in terms of a coefficient which gives the viscosity in the A_1-phase, and the result for temperatures close to T_{c1} and at melting pressure is $\frac{\delta\eta}{\eta(T_{c1})} = -(2.7 \pm 0.2)(1 - \frac{T}{T_{c1}})^{1/2}$. This formula also fits with the data of Roobol *et al* [3]. Our results for $P = 21$ bar are $\frac{\delta\eta_{xy}}{\eta_{xy}} = -2.09\tilde{\Delta}_{\max}$ and $\frac{\delta\eta_{zx}}{\eta_{zx}} = -\frac{4}{5}(2.09)\tilde{\Delta}_{\max}$, and for the melting pressure we have $\frac{\delta\eta_{xy}}{\eta_{xy}} = -1.92\tilde{\Delta}_{\max}$ and $\frac{\delta\eta_{zx}}{\eta_{zx}} = -\frac{4}{5}(1.92)\tilde{\Delta}_{\max}$. One expects that in the experiments the measured viscosity is $(\eta_{xy} + \eta_{zx})/2$ [6], hence we have $\frac{\delta\eta}{\eta} = -3.30(1 - \frac{T}{T_{c1}})^{1/2}$, $\frac{\delta\eta}{\eta} = -3.05(1 - \frac{T}{T_{c1}})^{1/2}$ for pressures of 21 and 34.36 bar respectively. We therefore see that agreement between our results and the experiments is fairly good.

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